CHPP surfaces: a new construction method for surfaces of general type with $p_g = q = 2$ (joint work with Fabrizio Catanese)

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Birational classification = Classification of minimal models

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- $\chi := \chi(S) = 1 - q + p_g$ *holomorphic Euler-Poincaré characteristic*

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- $K^2 \leq 9\chi$ (Bogomolov-Miyaoka-Yau)
- $K^2 \geq 2p_g$ if $q > 0$ (Debarre).

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If $\chi = 1$ even constructing examples of such surfaces is really hard!

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Remark

A is an abelian variety of dimension $q(S) = 2$.

The Construction of CHPP Surfaces

Let A' be an abelian surface with a divisor D yielding a polarization of type $(1, 2)$ and set

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The kernel $\mathcal{K}(D)$ of this map is

$$\mathcal{K}(D) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

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$\mathcal{K}(D) \cong \mathcal{H}_D/\mu_2$ acts on $\mathbb{P}^1 \times A' := \mathbb{P}(V) \times A'$ (action of product type)

$$S'_\lambda := \{x_1(y_1^3 + \lambda y_1 y_2^2) + x_2(y_2^3 + \lambda y_2 y_1^2) = 0\} \subset \mathbb{P}^1 \times A', \quad \lambda \in \mathbb{C}$$

- $\{x_1, x_2\}$ is a canonical basis of $V := H^0(A', D)$
- $\{y_1, y_2\}$ homogeneous coordinates of $\mathbb{P}^1 := \mathbb{P}(V)$,

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Definition

The family of surfaces $\{S_\lambda\}_{\lambda \in U \subset \mathbb{C}}$ is called the family of *CHPP surfaces*.

The Component of CHPP Surfaces

Theorem (-, Catanese)

The CHPP surfaces yields a unirational irreducible connected component of the moduli space of surfaces of general type with $K_S^2 = 5$, $p_g(S) = q(S) = 2$, and Albanese map $\alpha : S \to A = Alb(S)$ of degree $d = 3$.

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Remark

This component contains the family of surfaces constructed by Alfred Jungkai Chen and Christopher Derek Hacon in 2006 and coincides with the component found by Matteo Penegini and Francesco Polizzi in 2013.

$p_g = q = 2$: *PP4 surfaces*

PP4 surfaces: $p_g(S) = q(S) = 2,\ K_S^2 = 6,\ d = 4$
Named after Penegini-Polizzi (known family)
S is the free quotient

$$S := S'/\mathcal{K}(D), \qquad \mathcal{K}(D) \cong (\mathbb{Z}/3\mathbb{Z})^2,$$

$$S' := S'(\mu) := \{\operatorname{rank}(M) \leq 1\} \subset \mathbb{P}^2 \times A',$$

$$M = \begin{pmatrix} x_1 & x_3 & x_2 \\ y_1^2 + \mu y_2 y_3 & y_3^2 + \mu y_1 y_2 & y_2^2 + \mu y_1 y_3 \end{pmatrix}, \quad \mu \in \mathbb{C}$$

- A' is an Abelian surface with a polarization D of type $(1, 3)$,
- $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$,
- $\{y_1, y_2, y_3\}$ homogeneous coordinates of $\mathbb{P}^2 = \mathbb{P}(V)$,

Theorem (-,Catanese)

The family of PP4 surfaces provides an irreducible, connected component of the moduli space of surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 4.
It coincides with the component found by Penegini-Polizzi in 2014.

$p_g = q = 2$: *the new family of AC3 surfaces*

AC3 surfaces: $p_g(S) = q(S) = 2,\ K_S^2 = 6,\ d = 3$

It is a new family with these invariants.

S is the free quotient

$$S := S'/\mathcal{K}(D), \qquad \mathcal{K}(D) \cong (\mathbb{Z}/3\mathbb{Z})^2,$$

$$S' := \{(y,z) \in \mathbb{P}^2 \times A' \mid \sum_{j=1}^{3} y_j x_j(z) = 0, \sum_{i=1}^{3} y_i^2 y_{i+1} = 0\} \subset \mathbb{P}^2 \times A',$$

- A' is an Abelian surface with a polarization D of type $(1,3)$,
- $\{x_1, x_2, x_3\}$ is a canonical basis of $V = H^0(A', \mathcal{O}_{A'}(D))$,
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Theorem (Catanese, Sernesi)

The family of AC3 surfaces provides a new component of the moduli space of surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 3.

Thanks for listening!